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Universality of nonperturbative effects in $c < 1$ noncritical string theory

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Abstract: Nonperturbative effects in $c < 1$ noncritical string theory are studied using the two-matrix model. Such effects are known to have the form fixed by the string equations but the numerical coefficients have not been calculated so far. Using the method proposed recently, we show that it is possible to determine the coefficients for (p, q) string theory. We find that they are indeed finite in the double scaling limit and universal in the sense that they do not depend on the details of the potential of the two-matrix model.

Keywords: D-branes, 2D Gravity, Matrix Models.

Contents

1. Introduction

The nonperturbative effects in string theory reveal one of the most stringy features of string theory. From the large-order behavior in perturbation series [1], or the calculation of the D-instanton effects [2], one can see that the nonperturbative effects are of the form $\exp(-S_0/g_\mathrm{s})$, where g_s is the string coupling constant. These are quite different from the nonperturbative effects for a point particle theory which are of the form $\exp(-S_0/g^2)$ in terms of the coupling constant g. Studying such effects is important because we may be able to get some clues about the nonperturbative formulation of string theory by doing so. Noncritical string theory is a useful toy model for such an investigation. It possesses much fewer degrees of freedom compared with the critical one, and it can be nonperturbatively defined via a matrix model. Nevertheless it has many features in common with critical string theories and we can get insight about critical ones by studying the noncritical ones.

Nonperturbative effects in noncritical string theory have been much studied. Especially in [3−5], the value of S_0 is derived from the string equation, or as the action of solitonic excitations. More recently, nonperturbative effects in the $c = 0$ noncritical string theory are analyzed concretely using the one-matrix model [6]. In [6], the next to leading order contributions to the nonperturbative effects, which can be identified with the chemical potential of D-instantons, were computed by the method of orthogonal polynomials. It was shown that the nonperturbative effects up to this order are universal in the sense that these were independent of details of the matrix model potential. Since one cannot fix the exact value of the next to leading contribution from the string equation, it implies that the string equation fails to include some information of the matrix model. This result was further discussed and generalized in [7−9].

A natural question is whether the results in [6] can be generalized to other noncritical string theories. Since (p,q) noncritical string theory is defined in a nonperturbative manner by taking an appropriate double scaling limit [10 − 12] of the two-matrix model [13 − 15], it is necessary to examine nonperturbative effects by using the two-matrix model. This is the problem we address in this paper.

For this purpose, we first define the effective potential of the matrix eigenvalues as in [6]. The nonperturbative effects are due to the stationary points of this effective potential. In [16] the leading order contributions to the nonperturbative effects are obtained from this effective potential for the two-matrix model and are shown to coincide with the results in the continuum approach given in [17]. (Each stationary point corresponds to various ZZ-brane [18].) What we would like to do is to calculate the next to leading order contribution. In order to do this, we generalize the method proposed recently in [19] to the two-matrix model case. We further prove that the result is universal in the double scaling limit as in the $c = 0$ case.

This paper is organized as follows. In section 2, we define the effective potential of the matrix eigenvalues for the two-matrix model. The nonperturbative corrections to the free energy can be expressed in terms of this potential. In section 3, we summarize some of the known facts about the two-matrix model which will be used in section 4 and appendix B. In section 4, we calculate the nonperturbative effects up to the next leading order, and find that they are universal in the double scaling limit. Section 5 is devoted to conclusions and discussions. In appendix A we present a result of the chemical potentials in the case of higher critical points of the one-matrix model in order to compare with our results for the two-matrix model. In appendix B details of the computation of the denominator in the definition of the chemical potential are given.

2. Nonperturbative effects in $c < 1$ noncritical string theory

The noncritical string theories for $c < 1$ can be studied by using the two-matrix model,

$$\int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left\{-\frac{N}{g}\mathrm{Tr}\left[U(\mathbf{X}) + \widetilde{U}(\mathbf{Y}) - \mathbf{X}\mathbf{Y}\right]\right\}. \tag{2.1}$$

Here, $\mathbf{X}, \mathbf{Y}$ are $N \times N$ hermitean matrices, and $U(\mathbf{X}), \widetilde{U}(\mathbf{Y})$ are polynomials of $\mathbf{X}, \mathbf{Y}$, respectively. This matrix integral can be expressed as an integral over the eigenvalues of $\mathbf{X}, \mathbf{Y}$:

$$\int \prod_{i=1}^{N} \mathrm{d}x_i \mathrm{d}y_i \triangle(x)\triangle(y) \exp\left\{-\frac{N}{g}\sum_{i=1}^{N}\left[U(x_i) + \widetilde{U}(y_i) - x_i y_i\right]\right\}, \tag{2.2}$$

where $\triangle(z) = \prod_{i>j}(z_i - z_j)$ is the Vendermonde determinant.

In order to study the nonperturbative effects, it is convenient to define the effective potential $V_{\text{eff}}(x,y)$ of the matrix eigenvalues, as in the one-matrix case [6]. Picking up the N-th eigenvalue (x_N, y_N), we represent it as (x,y). Then, integrating over the other eigenvalues, we obtain

$$\int \prod_{i=1}^{N-1} \mathrm{d}x_i'\mathrm{d}y_i' \triangle(x')\triangle(y') \exp\left\{-\frac{N}{g}\sum_{i=1}^{N-1}\left[U(x_i')+\widetilde{U}(y_i')-x_i'y_i'\right]\right\} \times$$
$$\times \prod_{i=1}^{N-1}(x-x_i')(y-y_i')\exp\left\{-\frac{N}{g}\left[U(x)+\widetilde{U}(y)-xy\right]\right\}. \tag{2.3}$$

This quantity is regarded as the Boltzmann weight for (x, y). Rewriting eq. (2.3) in terms of $(N-1)\times(N-1)$ hermitean matrices $\mathbf{X}', \mathbf{Y}'$, we can define $V_{\mathrm{eff}}(x, y)$ in the following manner:

$$\exp\left[-V_{\mathrm{eff}}(x,y)\right] = \tag{2.4}$$
$$= \frac{\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}\det(x-\mathbf{X}')\det(y-\mathbf{Y}')}{\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}},$$

where $g' = (1-1/N)g$.

To obtain $V_{\mathrm{eff}}(x, y)$, one may calculate the right-hand side of eq. (2.4) as

$$\exp\left\{-\frac{N-1}{g'}\left[U(x)+\widetilde{U}(y)-xy\right]+\left\langle \mathrm{Tr}\log(x-\mathbf{X}')\right\rangle+\left\langle \mathrm{Tr}\log(y-\mathbf{Y}')\right\rangle+\right.$$
$$+\frac{1}{2}\left\langle \mathrm{Tr}\log(x-\mathbf{X}')\mathrm{Tr}\log(x-\mathbf{X}')\right\rangle_{\mathrm{c}}+\frac{1}{2}\left\langle \mathrm{Tr}\log(y-\mathbf{Y}')\mathrm{Tr}\log(y-\mathbf{Y}')\right\rangle_{\mathrm{c}}+$$
$$\left.+\left\langle \mathrm{Tr}\log(x-\mathbf{X}')\mathrm{Tr}\log(y-\mathbf{Y}')\right\rangle_{\mathrm{c}}+\cdots\right\}, \tag{2.5}$$

where

$$\left\langle F(\mathbf{X}',\mathbf{Y}')\right\rangle = \frac{\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}F(\mathbf{X}',\mathbf{Y}')}{\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}}, \tag{2.6}$$

and the subscript c denotes the connected part. Since insertions of $\mathrm{Tr}\log(x-\mathbf{X}')$ and $\mathrm{Tr}\log(y-\mathbf{Y}')$ correspond to boundaries (or loops) on the worldsheet, eq.(2.5) can be considered as a genus expansion of the free energy in open string theory.

However, this expansion is not always valid. In the one-matrix case, as was discussed in [19], such an expansion is not valid when the argument of V_{eff} is in the region where other eigenvalues are distributed. For such an argument, the effect of the repulsive potential between two eigenvalues cannot be treated perturbatively in $1/N$. This argument is also valid in the two-matrix case, and we cannot trust the expansion eq. (2.5) for x or y in the region where other x's or y's are distributed. Therefore we should calculate the effect of the eigenvalues in these regions using some other techniques. On the other hand, as we will see in the next section, the nonperturbative effects are due to the isolated saddle points for which x and y are outside such regions. Therefore we can use eq. (2.5) to study the fluctuations around these saddle points. The situation is quite the same as that for the one-matrix model and we can get the nonperturbative correction to the free energy due to the saddle point (x, y) as [6]

$$\delta F = N \frac{\int_{(x,y)} \mathrm{d}x' \mathrm{d}y' \exp\left[-V_{\mathrm{eff}}(x', y')\right]}{\int \mathrm{d}x' \mathrm{d}y' \exp\left[-V_{\mathrm{eff}}(x', y')\right]}. \tag{2.7}$$

Here, the symbol $\int_{(x,y)} \mathrm{d}x'\mathrm{d}y'$ means that the integral in the numerator is evaluated around (x, y) by the saddle point method. The numerator corresponds to the configuration where one set of eigenvalues is sitting at the saddle point (x, y). If this saddle point survives in the double scaling limit, the corresponding configuration would be regarded as a D-instanton configuration. In the double scaling limit, δF becomes of the form

$$\exp\left[-\frac{S_0}{g_{\mathrm{s}}} + \log \mu_{\mathrm{inst.}} + \mathcal{O}(g_{\mathrm{s}})\right]. \tag{2.8}$$

Here, S_0 and $\mu_{\mathrm{inst.}}$ can be considered as the classical action and the chemical potential of the D-instanton, respectively. The main purpose of this paper is to calculate δF in the double scaling limit and see if it is a universal quantity.

3. The two-matrix model

For the calculation of the numerator of eq. (2.7), we can use eq. (2.5). Using this formula, we obtain the leading order contribution to $V_{\mathrm{eff}}(x, y)$ as

$$V_{\mathrm{eff}}^{(0)}(x, y) = \frac{N-1}{g'}\left[U(x) + \widetilde{U}(y) - xy\right] - \left\langle \mathrm{Tr}\log(x - \mathbf{X}')\right\rangle - \left\langle \mathrm{Tr}\log(y - \mathbf{Y}')\right\rangle. \tag{3.1}$$

We can evaluate this from the large-N limit of the resolvents,

$$W(x) = \frac{1}{N-1}\left\langle \mathrm{Tr}\frac{1}{x - \mathbf{X}'}\right\rangle, \qquad \widetilde{W}(y) = \frac{1}{N-1}\left\langle \mathrm{Tr}\frac{1}{y - \mathbf{Y}'}\right\rangle. \tag{3.2}$$

For convenience, we rather consider the following combinations,

$$Y(x) = U'(x) - g'W(x), \qquad X(y) = \widetilde{U}'(y) - g'\widetilde{W}(y) .. \tag{3.3}$$

The functions $Y(x)$ and $X(y)$ have been much studied in the literature (see [20] and references therein). Here we summarize some known properties of $Y(x)$ and $X(y)$ for later convenience. Note first that as a function of x (y) on the complex plane, $Y(x)$ $(X(y))$ has a cut on the real axis, where the eigenvalues of $\mathbf{X}'$ $(\mathbf{Y}')$ are distributed. One can define the Riemann surface of $Y(x)$ $(X(y))$. The complex plane we started from is called the physical sheet of x (y). In other words, the physical sheet of x (y) is the sheet that contains the infinity $x = \infty_x$ $(y = \infty_y)$ around which $Y(x)$ $(X(y))$ is expanded as

$$Y(x) \sim U'(x) - \frac{g'}{x} + \mathcal{O}\left(x^{-2}\right) \quad \left(X(y) \sim \widetilde{U}'(y) - \frac{g'}{y} + \mathcal{O}(y^{-2})\right). \tag{3.4}$$

Actually the functions $Y(x)$ and $X(y)$ satisfy

$$y = Y(x), \qquad x = X(y), \tag{3.5}$$

which can be proved by showing that the pairs $(x, Y(x))$ and $(X(y), y)$ satisfy the same algebraic equation. This relation implies that the Riemann surfaces of $Y(x)$ and $X(y)$ are actually the same. As usual, eq. (3.5) is solved for x, y as $x = \mathcal{X}(s)$, $y = \mathcal{Y}(s)$. Here, $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are certain functions of a uniformization parameter s, which globally parametrizes the Riemann surface of eq. (3.5). Since we are interested in (p, q) noncritical string theory, we can restrict ourselves to the Riemann surface with genus zero. In this case, the uniformization parameter s takes values in $C \cup \infty$, and the functions $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are known to be expanded as

$$\mathcal{X}(s) = \gamma s + \sum_{k=0}^{\widetilde{d}-1} \frac{\alpha_k}{s^k}\,, \qquad \mathcal{Y}(s) = \frac{\gamma}{s} + \sum_{k=0}^{d-1} \beta_k s^k\,, \tag{3.6}$$

where d and $\widetilde{d}$ are the degrees of the potentials $U(x)$ and $\widetilde{U}(y)$, respectively. Here, the constants α_k, β_k and γ are determined from $U(x)$, $\widetilde{U}(y)$ and g. In terms of $\mathcal{X}(s)$ and $\mathcal{Y}(s)$, we can express $Y(x)$ and $X(y)$ as

$$Y(x) = \mathcal{Y}(\mathcal{X}^{-1}(x))\,, \qquad X(y) = \mathcal{X}(\mathcal{Y}^{-1}(y))\,. \tag{3.7}$$

Let us note that

$$\lim_{s\to\infty} \mathcal{X}(s) = \infty_x\,, \qquad \lim_{s\to 0} \mathcal{Y}(s) = \infty_y\,. \tag{3.8}$$

These equations mean that $x = \infty_x$ corresponds to $s = \infty$ and $y = \infty_y$ corresponds to $s = 0$.

Once the functions $Y(x)$ and $X(y)$ are given as in eq.(3.7), we obtain $V_{\text{eff}}^{(0)}(x, y)$ as

$$\begin{aligned} \frac{N-1}{g'} \lim_{\Lambda_x\to\infty_x} \lim_{\Lambda_y\to\infty_y} \Bigg[\int_{\Lambda_x}^{x} \mathrm{d}x' Y(x') + \int_{\Lambda_y}^{y} \mathrm{d}y' X(y') - xy + \\ + U(\Lambda_x) + U(\Lambda_y) - g' \log(\Lambda_x \Lambda_y) \Bigg]\,. \end{aligned} \tag{3.9}$$

In the following, whenever Λ_x or Λ_y appears, it is understood that we take the limit $\lim_{\Lambda_x\to\infty}$ or $\lim_{\Lambda_y\to\infty}$ and we will not write these symbols explicitly. We can also express $V_{\text{eff}}^{(0)}(x, y)$ in terms of the single-valued functions $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ on the Riemann surface. Indeed, by changing the variables x and y as $x = \mathcal{X}(s)$ and $y = \mathcal{Y}(\tilde{s})$, we obtain

$$\begin{aligned} V_{\text{eff}}^{(0)}(x, y) = \frac{N-1}{g'} \Bigg[\int_{\Lambda_x/\gamma}^{s} \mathrm{d}s' \mathcal{Y}(s') \partial \mathcal{X}(s') + \int_{\gamma/\Lambda_y}^{\tilde{s}} \mathrm{d}s' \mathcal{X}(s') \partial \mathcal{Y}(s') - \mathcal{X}(s) \mathcal{Y}(\tilde{s}) + \\ + U(\Lambda_x) + \widetilde{U}(\Lambda_y) - g' \log(\Lambda_x \Lambda_y) \Bigg]\,. \end{aligned} \tag{3.10}$$

From the expression eq. (3.10), we can see that the potential $V_{\text{eff}}^{(0)}(x, y)$ has local extrema at (x, y) with x and y satisfying the saddle point equations,

$$x = \mathcal{X}(s) = \mathcal{X}(\tilde{s})\,, \qquad y = \mathcal{Y}(s) = \mathcal{Y}(\tilde{s})\,. \tag{3.11}$$

The saddle points solving eq. (3.11) are divided into two classes. One is of the trivial saddle points (x, y) satisfying $s = \tilde{s}$. The other is of the non-trivial saddle points (x, y) satisfying $s \neq \tilde{s}$. It is not the former class but the latter class of saddle points, which can

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contribute to the nonperturbative corrections to the free energy [16]. Thus, we consider only the latter class of saddle points below. The non-trivial saddle points are nothing but the double points, or the singularities, of the Riemann surface [21].

4. The chemical potential of D-instantons

In order to get the chemical potential of D-instantons, we should consider the next to leading order contribution in the large-N limit. From eq. (2.5) we can see that we need two loop correlators in the two-matrix model. It is known that these correlators can be expressed in terms of the functions $\mathcal{X}(s)$, $\mathcal{Y}(s)$ as follows [22]:

$$\begin{aligned}\left\langle \mathrm{Tr}\log(x-\mathbf{X}')\mathrm{Tr}\log(x'-\mathbf{X}')\right\rangle_c &= -\log\frac{\mathcal{X}(s)-\mathcal{X}(s')}{s-s'}+\log\gamma\,,\\ \left\langle \mathrm{Tr}\log(y-\mathbf{Y}')\mathrm{Tr}\log(y'-\mathbf{Y}')\right\rangle_c &= -\log\frac{\mathcal{Y}(\tilde{s})-\mathcal{Y}(\tilde{s}')}{1/\tilde{s}-1/\tilde{s}'}+\log\gamma\,,\\ \left\langle \mathrm{Tr}\log(x-\mathbf{X}')\mathrm{Tr}\log(y-\mathbf{Y}')\right\rangle_c &= -\log\left(1-\frac{\tilde{s}}{s}\right).\end{aligned} \tag{4.1}$$

Here, x,x',y,y' are related to $s,s',\tilde{s},\tilde{s}'$ as $x=\mathcal{X}(s),\ x'=\mathcal{X}(s'),\ y=\mathcal{Y}(\tilde{s}),\ y'=\mathcal{Y}(\tilde{s}')$.

Using these correlators, we can evaluate $V_{\rm eff}(x',y')$ for the point $(x',y')=(\mathcal{X}(s'),\mathcal{Y}(\tilde{s}'))$ near the double point $(x,y)=(\mathcal{X}(s),\mathcal{Y}(\tilde{s}))$ as

$$\begin{aligned}V_{\rm eff}(x',y') = {}& \frac{N-1}{g'}V^{(0)}_{\rm eff}(x',y')+\\ &+\frac{1}{2}\log\left[\partial\mathcal{X}(s')\right]+\frac{1}{2}\log\left[-\tilde{s}'^2\partial\mathcal{Y}(\tilde{s}')\right]-\log\gamma+\log\left(1-\frac{\tilde{s}'}{s'}\right)+\\ &+\mathcal{O}\left(\frac{1}{N}\right),\end{aligned} \tag{4.2}$$

where $V^{(0)}_{\rm eff}(x',y')$ is the potential calculated from eq. (3.10). Now, it is straightforward to evaluate the numerator of eq. (2.7). The result is

$$\begin{aligned}&\int_{(x,y)}\mathrm{d}x'\mathrm{d}y'\exp\left[-V_{\rm eff}(x',y')\right]\simeq\\ &\quad\simeq\gamma\left(1-\frac{\tilde{s}}{s}\right)^{-1}\tilde{s}^{-1}\left[-\partial\mathcal{X}(s)\partial\mathcal{Y}(\tilde{s})\right]^{-1/2}\left(\frac{2\pi g'}{N-1}\right)\left[\frac{\partial\mathcal{Y}(s)}{\partial\mathcal{X}(s)}\frac{\partial\mathcal{X}(\tilde{s})}{\partial\mathcal{Y}(\tilde{s})}-1\right]^{-1/2}\times\\ &\quad\times\exp\left\{\frac{N-1}{g'}\left[2\mathrm{R}-\int_{\tilde{s}}^{s}\mathrm{d}s'\mathcal{Y}(s')\partial\mathcal{X}(s')\right]\right\},\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}2\mathrm{R} &= \int_{\gamma/\Lambda_y}^{\Lambda_x/\gamma}\mathrm{d}s'\mathcal{Y}(s')\partial\mathcal{X}(s')+\mathcal{X}\left(\frac{\gamma}{\Lambda_y}\right)\mathcal{Y}\left(\frac{\gamma}{\Lambda_y}\right)-U(\Lambda_x)-\widetilde{U}(\Lambda_y)+g'\log(\Lambda_x\Lambda_y)\\ &= \int_{X(\Lambda_y)}^{\Lambda_x}\mathrm{d}x'Y(x')+X(\Lambda_y)\Lambda_y-U(\Lambda_x)-\widetilde{U}(\Lambda_y)+g'\log(\Lambda_x\Lambda_y)\,.\end{aligned} \tag{4.4}$$

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The denominator of eq. (2.7) can be calculated as follows [19]. First, let us note that the denominator of eq. (2.7) is expressed as

$$\int \mathrm{d}x\mathrm{d}y \exp\left[-V_{\text{eff}}(x,y)\right] = \tag{4.5}$$
$$= \frac{\int \mathrm{d}x\mathrm{d}y \int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}\det(x-\mathbf{X}')\det(y-\mathbf{Y}')}{\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}}.$$

Since the numerator of eq. (4.6) is proportional to the partition function

$$\int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left\{-\frac{N}{g}\mathrm{Tr}\left[U(\mathbf{X})+\widetilde{U}(\mathbf{Y})-\mathbf{X}\mathbf{Y}\right]\right\}, \tag{4.6}$$

what we should calculate is essentially the ratio between the matrix integrals over $(\mathbf{X},\mathbf{Y})$ and over $(\mathbf{X}',\mathbf{Y}')$. After some calculations, which are presented in the appendix B, we obtain

$$\int \mathrm{d}x\mathrm{d}y \exp\left[-V_{\text{eff}}(x,y)\right] \simeq (2\pi)^{3/2}\gamma\sqrt{(N-1)g'}\exp\left[\frac{2(N-1)\mathrm{R}}{g'}\right]. \tag{4.7}$$

From eq. (4.3) and eq. (4.7), we get

$$\delta F \simeq \sqrt{\frac{g'}{2\pi(N-1)}}\frac{(1-\tilde{s}/s)^{-1}\,\tilde{s}^{-1}}{\left[\partial\mathcal{X}(s)\partial\mathcal{Y}(\tilde{s})-\partial\mathcal{X}(\tilde{s})\partial\mathcal{Y}(s)\right]^{1/2}}$$
$$\times\exp\left[-\frac{N-1}{g'}\int_{\tilde{s}}^{s}\mathrm{d}s'\mathcal{Y}(s')\partial\mathcal{X}(s')\right]. \tag{4.8}$$

It is possible to express eq. (4.8) in terms of $\mathcal{X}(s)$ and $Y(x)=\mathcal{Y}(\mathcal{X}^{-1}(x))$. By doing so, one finds that δF depends not on $Y(x)$ itself but on the non-polynomial part, or the singular part, of $Y(x)$. Next, we will use this fact to evaluate δF in the double scaling limit.

We need to know how $\mathcal{X}(s)$ and the singular part of $Y(x)$ scale as we take the double scaling limit which realizes (p,q) noncritical string theory. Such a scaling limit is described in the following way [22]. Let us introduce a positive cut-off parameter $a \propto N^{-1/(p+q)}$ and consider the limit $a \to 0$ $(N\to\infty)$. In order to describe the limit, let us parametrize s by a new uniformization parameter ω as $s = s_* \exp(a\xi\omega)$, where s_* is the would-be critical point and ξ is a positive parameter, whose power ξ^{2p} should be identified with the cosmological constant. Now, we take the limit $a \to 0$ while we tune the potentials $U(x)$ and $\widetilde{U}(y)$ to realize (p,q) noncritical string theory. Such a fine tuning can be carried out so that the leading order contribution to $x=\mathcal{X}(s)$ and the singular part of $Y(x)$ are given as

$$x - x_* \sim 2Ca^p\xi^p\mathrm{T}_p(\omega)\,,\qquad Y_{\text{sing}}(x)-Y_{\text{sing}}(x_*) \sim 2\widetilde{C}a^q\xi^q\mathrm{T}_q(\omega)\,, \tag{4.9}$$

where $x_* = \mathcal{X}(s_*)$ and $\mathrm{T}_n(\cosh\theta)=\cosh(n\theta)$ is the Chebyshev polynomial of the first kind. Here, C and $\widetilde{C}$ are certain constants.

Let us introduce new scaling variables $\hat{x}$ and $\hat{y}$ as

$$x - x_* = 2Ca^p\xi^p\hat{x}\,,\qquad Y_{\text{sing}}(x)-Y_{\text{sing}}(x_*) = 2\widetilde{C}a^q\xi^q\hat{y}\,. \tag{4.10}$$

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Then, we can see that in the double scaling limit, the Riemann surface of eq. (3.5) is reduced to that of

$$\mathrm{T}_q(\hat{x}) - \mathrm{T}_p(\hat{y}) = 0\,. \tag{4.11}$$

In this way, we reach the Riemann surface associated with (p, q) noncritical string [21]. As was shown in [21], the Riemann surface of (4.11) is again with genus zero and has $(p-1)(q-1)/2$ inequivalent double points. To describe the Riemann surface of eq. (4.11) and its double points, let us parametrize ω as $\omega = \cosh\theta$. In this parametrization, we have

$$\hat{x} = \cosh(p\theta)\,, \qquad \hat{y} = \cosh(q\theta)\,. \tag{4.12}$$

Thus, we can conclude that in the double scaling limit, the double points are given by the pairs, $(\theta_{m,-n}, \theta_{m,n})$, where

$$\theta_{m,n} = i\pi\left(\frac{m}{p} + \frac{n}{q}\right), \tag{4.13}$$

with $1 \leq m \leq p-1,\ 1 \leq n \leq q-1$ and $mq - np > 0$ [21].

Now it is not difficult to evaluate δF in the double scaling limit. By putting all ingredients together and taking $N \to \infty, a \to 0$ with

$$N|C\widetilde{C}|g^{-1}a^{p+q} = g_{\mathrm{s}}^{-1}\,, \tag{4.14}$$

fixed, we obtain

$$\begin{aligned}
\delta F \;=\; & \frac{1}{8}\sqrt{\frac{g_{\mathrm{s}}}{2\pi pq\xi^{p+q}}}\left(\sin\frac{\pi m}{p}\sin\frac{\pi n}{q}\right)^{-1} \times \\
& \times \left(\cos\frac{2\pi m}{p} - \cos\frac{2\pi n}{q}\right)^{1/2}\left[\eta(-1)^{m+n}\sin\frac{\pi mq}{p}\sin\frac{\pi np}{q}\right]^{-1/2} \times \\
& \times \exp\left[-\eta(-1)^{m+n}\frac{8pq}{g_{\mathrm{s}}(q^2-p^2)}\xi^{p+q}\sin\frac{\pi mq}{p}\sin\frac{\pi np}{q}\right].
\end{aligned} \tag{4.15}$$

Here η is the sign of $C\widetilde{C}$, which can be determined as $\eta = \mathrm{sign}(\sin(\pi q/p))$. This follows from the condition that the distribution function of eigenvalues of $\mathbf{X}'$, defined by

$$\rho_x(x) = -\frac{1}{\pi}\mathrm{Im}\,W(x + \mathrm{i}0) = \frac{1}{\pi g'}\mathrm{Im}\,Y(x + \mathrm{i}0)\,, \tag{4.16}$$

satisfies $\rho_x(x) \geq 0$. Here, x was understood to be on the physical sheet.[1]

Thus we have obtained δF in the double scaling limit. Essentially it depends only on the combination ξ^{p+q}/g_{s} which is the scaling variable in the continuum theory. Therefore we have shown that δF to this order is a universal quantity. As a check of the universality, we can compare the result with that of the one matrix model. One can realize the $(2, 2k+1)$ noncritical string theory as the higher critical point of the one-matrix model. It is possible to calculate δF for such string theory using the results in [19]. As we will show in appendix A, the result perfectly agrees with eq. (4.15).

[1]In the parametrization $x \propto \cosh p\theta$, the physical sheet of x is the domain of $0 < |\mathrm{Im}\theta| < \pi/p$ [16].

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Here we have identified the double points $(s, \tilde{s})$ in eq. (4.8) with $(\theta_{m,-n}, \theta_{m,n})$ via $s = s_* \exp(a\xi \cosh\theta)$. In principle, one can also consider the double points $(\theta_{m,n}, \theta_{m,-n})$, whose classical action S_0 has the sign opposite to the one in eq. (4.15). Our choice is a natural one such that the action part S_0 agrees with the one proposed in the continuum theory [23], with η given above.

5. Conclusions and discussions

In this paper we generalize the result in [6] to the two-matrix model case and study the nonperturbative effects for (p, q) noncritical string theory. Utilizing the method given in [19], we can obtain the nonperturbative effects in the form of $\exp[-S_0/g_{\rm s} + \log \mu_{\rm inst}]$. We find that S_0 reproduces the known results and that $\mu_{\rm inst}$ is finite and universal as in the case of the $c = 0$ noncritical string theory. Although there are many non-universal parameters involved, the final result depends only on the scaling variable $\xi^{p+q}/g_{\rm s}$.

Although we get the number, the physical meaning of δF is not obvious for most of (p, q) and (m, n). For some combination δF becomes imaginary and we can see that it is related to some instability. However for the two-matrix model, we know the behavior of the effective action $V_{\rm eff}(x, y)$ only around the saddle points, and we cannot get the clear picture of such instabilities.

It will be intriguing to study what we have obtained from the point of view of the loop equation or the string field theory. In [6], the nonperturbative effects of $c = 0$ string theory were studied from this point of view, although the calculation of the chemical potential itself seems difficult in such an approach. In a recent paper [24], the chemical potential for $(p, p+1)$ noncritical string theory was calculated by making some assumptions in the SFT approach proposed in [5]. Their results agree with ours eq. (4.15) up to a factor of i. It would be interesting to check if we can perform similar calculations using other string field theories [25, 26].

It would also be interesting to compute loop amplitudes in a fixed D-instanton background as was done in [6] for the $c = 0$ case. Comparing it with the Liouville results, we will be able to get a strong evidence for identifying the isolated eigenvalue as the D-instanton in $c < 1$ noncritical string theory. Such a study will be useful to understand what D-brane is in string theory.

We believe our work would be instructive to the study of nonperturbative effects of critical string theory. The proposed candidates of the nonperturbative formulation of critical string theory are, in sense, multi-matrix models [27−29]. Thus in order to go from $c = 0$ to critical string theory, it is indispensable to know how to extend the analysis in the case of the one-matrix model to a multi-matrix model.

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A. The $(2, 2k+1)$ models

The nonperturbative effects for the higher critical points of the one-matrix model can be studied in the same way as in the $c = 0$ case.[2].

In the one-matrix model, the resolvent can be obtained as

$$\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\right\rangle = \frac{1}{2g^2}\left[V'(z) - M(z)\sqrt{(z-\alpha)(z-\beta)}\right], \tag{A.1}$$

where $M(z)$ is a polynomial (see [19] for the notation). In the continuum limit corresponding to the $(2, 2k+1)$ model, we take $a \to 0$ with

$$\begin{aligned} z &\sim \beta + a[\cosh(2\theta) + 1]\,, \\ \frac{1}{g^2}M(z)\sqrt{(z-\alpha)(z-\beta)} &\sim C(-1)^k a^{k+1/2}\cosh[(2k+1)\theta]\sqrt{\beta-\alpha}\,, \end{aligned} \tag{A.2}$$

where C is a positive constant. Using eq. (A.2), we can evaluate $V_{\rm eff}^{(0)}(z)$, the large-N limit of the effective potential of the matrix eigenvalues [6] and see that the isolated saddle points of $V_{\rm eff}^{(0)}(z)$ are located at the positions corresponding to

$$\theta = \frac{\pi \mathrm{i}(2l+1)}{2(2k+1)}\,, \tag{A.3}$$

with $l = 0, 1, \cdots, k-1$. Then, using the results in [19], we can show that these saddle points contribute to the nonperturbative corrections to the free energy as

$$\begin{aligned} \delta F =& \frac{t^{-(2k+3)/8}}{4\sqrt{\pi}\left[1+\cos(\frac{2l+1}{2k+1}\pi)\right]}\left[(-1)^{l+k}\frac{\sin\left(\frac{2l+1}{2k+1}\pi\right)}{2k+1}\right]^{1/2} \times \\ &\times \exp\left[(-1)^{l+k}t^{(2k+3)/4}\frac{2(2k+1)}{(2k+1)^2-4}\sin\left(\frac{2l+1}{2k+1}\pi\right)\right], \end{aligned} \tag{A.4}$$

with t the cosmological constant. We can see that eq. (A.4) indeed agrees with eq. (4.15) by choosing $(m, n) = (1, k-l)$ and $\xi^{2k+3}/g_{\rm s} = t^{(2k+3)/4}/8$ in eq. (4.15).

B. The denominator

In this appendix, we will explain how to calculate $\int \mathrm{d}x\mathrm{d}y \exp[-V_{\rm eff}(x, y)]$ and get the result eq. (4.7) in the large-N limit.

For this purpose, it is convenient to define Z_N and Z'_{N-1} as

$$\begin{aligned} Z_N =& \int \prod_{i=1}^{N} \mathrm{d}x_i \mathrm{d}y_i \triangle(x)\triangle(y)\exp\left\{-\frac{N}{g}\sum_i\left[U(x_i) + \widetilde{U}(y_i) - x_i y_i\right])\right\}, \\ Z'_{N-1} =& \int \prod_{i=1}^{N-1} \mathrm{d}x'_i \mathrm{d}y'_i \triangle(x')\triangle(y')\exp\left\{-\frac{N-1}{g'}\sum_i\left[U(x'_i) + \widetilde{U}(y'_i) - x'_i y'_i\right]\right\}, \end{aligned} \tag{B.1}$$

[2]See [7], for the analysis generalizing [6].

where $g = g'(1+\frac{1}{N-1})$. Then, we can express $\int \mathrm{d}x\mathrm{d}y \exp[-V_{eff}(x,y)]$ as Z_N/Z'_{N-1}.

Z_N and Z'_{N-1} can be expressed as matrix integrals as

$$Z_N = C_N \int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left\{-\frac{N}{g}\mathrm{Tr}\left[U(\mathbf{X})+\widetilde{U}(\mathbf{Y})-\mathbf{X}\mathbf{Y}\right]\right\}, \tag{B.2}$$

$$Z'_{N-1} = C'_{N-1} \int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}, \tag{B.3}$$

where X, Y are $N \times N$ matrices and X', Y' are $(N-1)\times(N-1)$ matrices and C_N and C'_{N-1} are constants. Let us assume that the quadratic part of $U(x)$ and $\widetilde{U}(y)$ are $\frac{c_1}{2}x^2$ and $\frac{c_2}{2}y^2$ respectively and take the integration measure $\mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y}$ and $\mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}'$ so that

$$\begin{aligned} 1 &= \int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left[-\frac{N}{g}\mathrm{Tr}\left(\frac{c_1}{2}\mathbf{X}^2+\frac{c_2}{2}\mathbf{Y}^2-\mathbf{X}\mathbf{Y}\right)\right], \\ 1 &= \int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left[-\frac{N-1}{g'}\mathrm{Tr}\left(\frac{c_1}{2}\mathbf{X}'^2+\frac{c_2}{2}\mathbf{Y}'^2-\mathbf{X}'\mathbf{Y}'\right)\right]. \end{aligned} \tag{B.4}$$

Here we assume $c_1c_2 - 1 \neq 0$. The constants C_N and C'_{N-1} are determined by choosing the gaussian potentials $U(x) = \frac{c_1}{2}x^2, \widetilde{U}(y) = \frac{c_2}{2}y^2$. By using the method of orthogonal polynomials, one can show that these constants satisfy

$$\frac{C_N}{C'_{N-1}} \sim (2\pi)^{3/2}\sqrt{N-1}e^{-(N-1)}\frac{(g')^N}{(c_1c_2-1)^{N-1/2}}, \tag{B.5}$$

for large N.

Now we would like to evaluate Z_N/Z'_{N-1}. For large N, the matrix integral in eq. (2.1) can be written as

$$\int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left\{-\frac{N}{g}\mathrm{Tr}\left[U(\mathbf{X})+\widetilde{U}(\mathbf{Y})-\mathbf{X}\mathbf{Y}\right]\right\} = \exp\left[\frac{N^2}{g^2}F_0(g)+F_1(g)+\frac{g^2}{N^2}F_2(g)+\cdots\right]. \tag{B.6}$$

Thus, we obtain

$$\begin{aligned} \frac{Z_N}{Z'_{N-1}} &= \frac{C_N\int \mathrm{d}\mathbf{X}\mathrm{d}\mathbf{Y} \exp\left\{-\frac{N}{g}\mathrm{Tr}\left[U(\mathbf{X})+\widetilde{U}(\mathbf{Y})-\mathbf{X}\mathbf{Y}\right]\right\}}{C'_{N-1}\int \mathrm{d}\mathbf{X}'\mathrm{d}\mathbf{Y}' \exp\left\{-\frac{N-1}{g'}\mathrm{Tr}\left[U(\mathbf{X}')+\widetilde{U}(\mathbf{Y}')-\mathbf{X}'\mathbf{Y}'\right]\right\}} \\ &= \frac{C_N}{C'_{N-1}}\exp\left[\frac{N-1}{g'}\partial_{g'}F_0(g')+\frac{1}{2}\partial_{g'}^2F_0(g')+\mathcal{O}\left(\frac{1}{N}\right)\right]. \end{aligned} \tag{B.7}$$

Therefore, to determine the denominator of eq. (2.7), we need the derivatives of $F_0(g')$. $F_0(g')$ was calculated in [30, 31]. Here we will follow their method to get the derivatives of $F_0(g')$.

Let us consider what happens to $\partial_{g'}F_0(g')$ if we change the potential U as $U \to U+\delta U$ where δU does not involve the quadratic part. From eq. (B.6), we can show that the variation of $F_0(g')$ itself is calculated as

$$\begin{aligned} \delta F_0(g') &= \frac{g'}{N-1}\oint_{\infty_x}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\delta U(x)\left\langle \mathrm{Tr}\frac{1}{x-\mathbf{X}'}\right\rangle \\ &= -\oint_{\infty_x}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\delta U(x)Y(x). \end{aligned} \tag{B.8}$$

Thus, we obtain

$$\delta\left[\partial_{g'} F_0(g')\right] = -\oint_{\infty_x} \frac{\mathrm{d}x}{2\pi i} \delta U(x) \partial_{g'} Y(x) \,. \tag{B.9}$$

Let us notice that under the integral sign over x, $\partial_{g'} Y(x)$ means the derivative of Y with x fixed. We can indicate this by putting the subscript as $(\partial_{g'} Y)_x$. $(\partial_{g'} Y)_x$ is related to the derivative $(\partial_{g'} \mathcal{Y})_s$ with the uniformization parameter s fixed. Indeed, by setting $x = \mathcal{X}(s)$, we obtain

$$(\partial_{g'} Y)_x = (\partial_{g'} \mathcal{Y})_s - \frac{\partial Y}{\partial x} (\partial_{g'} \mathcal{X})_s \,. \tag{B.10}$$

Similarly, by setting $y = \mathcal{Y}(s)$, we obtain

$$(\partial_{g'} X)_y = (\partial_{g'} \mathcal{X})_s - \frac{\partial X}{\partial y} (\partial_{g'} \mathcal{Y})_s \,. \tag{B.11}$$

Using these identities, we obtain

$$\mathrm{d}x (\partial_{g'} Y)_x = -\mathrm{d}y (\partial_{g'} X)_y \,, \tag{B.12}$$

for $(x, y) = (\mathcal{X}(s), \mathcal{Y}(s))$. Since the singularities of the one form $\mathrm{d}x\, \partial_{g'} Y(x) = -\mathrm{d}y\, \partial_{g'} X(y)$ are only the poles at $x = \infty_x$ and $y = \infty_y$, it can be identified with $\mathrm{d}s/s$. Therefore, we get

$$\delta\left[\partial_{g'} F_0(g')\right] = -[\delta U(x)]_0 \,, \tag{B.13}$$

where $[\delta U(x)]_0$ is the coefficient of s^0 when we expand $\delta U(x)|_{x=\mathcal{X}(s)}$ near $s = \infty$, or $s = 0$. Now from the expression of $Y(x)$, $[\delta U(x)]_0$ can be rewritten as

$$[\delta U(x)]_0 = \left[\int_{\Lambda_x}^{x} \mathrm{d}x' \delta Y(x')\right]_0 + \delta U(\Lambda_x) \,, \tag{B.14}$$

or since $\delta X(y) \sim y^{-2}$ for $y \sim \infty_y$,

$$[\delta U(x)]_0 = \delta \left[\int_{\Lambda_x}^{X(\Lambda_y)} \mathrm{d}x' Y(x')\right] + \delta U(\Lambda_x) \,. \tag{B.15}$$

On the other hand, if we consider the variation of the quantity 2R (see eq.(4.4)) under $U \to U + \delta U$, we obtain

$$\delta(2\mathrm{R}) = \delta \left[\int_{X(\Lambda_y)}^{\Lambda_x} \mathrm{d}x' Y(x')\right] - \delta U(\Lambda_x) \,. \tag{B.16}$$

Therefore it coincides with $\delta[\partial_{g'} F_0(g')]$. Since we can treat the variation of $\widetilde{U}$ in the same way, we can conclude that 2R coincides with $\partial_{g'} F_0(g')$ up to some function $f'(g', c_1, c_2)$.

We can determine f' by considering the gaussian case. For the gaussian case, using

$$\mathcal{X}(s) = \sqrt{\frac{g'}{c_1 c_2 - 1}} \left(s + \frac{c_2}{s}\right), \qquad \mathcal{Y}(s) = \sqrt{\frac{g'}{c_1 c_2 - 1}} \left(\frac{1}{s} + c_1 s\right), \tag{B.17}$$

we get

$$\int_{X(\Lambda_y)}^{\Lambda_x} \mathrm{d}x' Y(x') - U(\Lambda_x) - \widetilde{U}(\Lambda_y) + X(\Lambda_y)\Lambda_y + g' \ln \Lambda_x \Lambda_y = -g' \left(\ln \frac{c_1 c_2 - 1}{g'} + 1 \right). \quad \text{(B.18)}$$

Since $\partial_{g'} F_0(g')$ should vanish for the gaussian case, we obtain

$$\partial_{g'} F_0(g') = 2\mathrm{R} + g' \left(\ln \frac{c_1 c_2 - 1}{g'} + 1 \right). \quad \text{(B.19)}$$

We can express $\partial_{g'}^2 F_0(g')$ by using γ in eq. (3.6). To show this, we further reduce the size of the matrices and define

$$Z''_{N-2} = C''_{N-2} \int \mathrm{d}\mathbf{X}''\mathrm{d}\mathbf{Y}'' \exp\left\{ -\frac{N}{g} \mathrm{Tr} \left[U(\mathbf{X}'') + \widetilde{U}(\mathbf{Y}'') - \mathbf{X}''\mathbf{Y}'' \right] \right\}, \quad \text{(B.20)}$$

for $(N-2) \times (N-2)$ matrices $\mathbf{X}'', \mathbf{Y}''$. Here C''_{N-2} is defined by

$$\begin{aligned} C''_{N-2} &\int \mathrm{d}\mathbf{X}''\mathrm{d}\mathbf{Y}'' \exp\left\{ -\frac{N}{g} \mathrm{Tr} \left[U(\mathbf{X}'') + \widetilde{U}(\mathbf{Y}'') - \mathbf{X}''\mathbf{Y}'' \right] \right\} = \\ &= \int \prod_{i=1}^{N-2} \mathrm{d}x''_i \mathrm{d}y''_i \triangle(x'')\triangle(y'') \exp\left\{ -\frac{N}{g} \sum_i \left[U(x''_i) + \widetilde{U}(y''_i) - x''_i y''_i \right] \right\}. \quad \text{(B.21)} \end{aligned}$$

Then we can express $Z_N Z''_{N-2}/(Z'_{N-1})^2$ in the large-N limit as

$$\frac{g'}{c_1 c_2 - 1} \exp\left[\partial_{g'}^2 F_0(g') \right]. \quad \text{(B.22)}$$

On the other hand, $Z_N Z''_{N-2}/(Z'_{N-1})^2$ can be evaluated by the orthogonal polynomial technique and can be expressed in the large-N limit as γ^2. Therefore we get

$$\exp\left[\frac{1}{2} \partial_{g'}^2 F_0(g') \right] = \gamma \sqrt{\frac{c_1 c_2 - 1}{g'}}. \quad \text{(B.23)}$$

Putting all these together we obtain

$$\frac{Z_N}{Z'_{N-1}} = (2\pi)^{3/2} \gamma \sqrt{(N-1)g'} \exp\left[\frac{2(N-1)\mathrm{R}}{g'} \right]. \quad \text{(B.24)}$$

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